Published by Institute of Physics Publishing for SISSA/ISAS

Received: *May 31, 2002*
Accepted: *September 19, 2002*

# From AdS/CFT correspondence to hydrodynamics

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Abstract: We compute the correlation functions of R-charge currents and components of the stress-energy tensor in the strongly coupled large-$N$ finite-temperature $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, following a recently formulated minkowskian AdS/CFT prescription. We observe that in the long-distance, low-frequency limit, such correlators have the form dictated by hydrodynamics. We deduce from the calculations the R-charge diffusion constant and the shear viscosity. The value for the latter is in agreement with an earlier calculation based on the Kubo formula and absorption by black branes.

Keywords: Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence, Thermal Field Theory.

## Contents



## 1. Introduction

Recently, the correspondence between conformal field theories (CFT) and string theories or supergravity on certain background has been under intense investigation [1]–[4]. The original and best studied example is the correspondence between the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in the large $N$ limit and the type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ space. In the strong coupling limit of the field theory, the string theory is reduced to classical supergravity, which allows one to calculate all field-theory correlation functions.

One of the first nontrivial predictions of the gauge theory/gravity correspondence was that for the entropy of the $\mathcal{N} = 4$ SYM theory at finite temperature [5]. According to gravity calculations, the entropy of this theory in the limit of large 't Hooft coupling is precisely 3/4 times the value in the zero coupling limit. However, so far there has been no independent check of this result from the field-theoretical side. Indeed, due to the lack of supersymmetry at finite temperature no non-renormalization theorem is expected to hold, while the strong coupling calculations are notoriously difficult.

A serious effort has been recently invested in an attempt to build a quantitatively predictive gravity dual for a realistic gauge theory. With this work as yet incomplete, one nevertheless has discovered many aspects resembling those in the theory of strong interactions within the framework of the AdS/CFT correspondence [6, 7, 8].



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In this paper, we use the hydrodynamic limit as a nontrivial test of the AdS/CFT correspondence at finite temperature. The basic idea is very simple: the long-distance, low-frequency behavior of any interacting theory at finite temperature must be described by the old-fashioned theory of fluid mechanics (hydrodynamics) [9]. This statement has not been, and perhaps can never be, rigorously proven for all theories, but is strongly supported by the ample physical intuition coming from our experience with macroscopic systems. As the result of its universal nature, hydrodynamics implies very precise constraints on the forms of the correlation functions (in Minkowski space) of conserved currents and components of the stress-energy tensor: basically, these correlators are fixed once a few transport coefficients are known [10, 11].

The reliance on Minkowski correlators, as opposed to euclidean (or Matsubara) Green's functions, is what separates this work from most of previous work on finite-temperature AdS/CFT correspondence. The benefit of looking at minkowskian correlators is that while at finite temperature the euclidean correlation functions decay exponentially at large separations, the Minkowski counterparts possess long-time non-exponential tails which have universal character.

We shall perform a check that in the finite-temperature $\mathcal{N} = 4$ SYM theory, the hydrodynamic forms of the two-point minkowskian correlators are correctly reproduced by gravity. This paper deals with the diffusive modes only (for their definition see section 2), which include the diffusion of R-charges and the shear mode corresponding to transverse velocity fluctuations. The treatment of longitudinal velocity fluctuations, or sound waves, is technically more complicated and is deferred to future work.

The paper is constructed as follows. In section 2 we briefly review hydrodynamics equations and their implications for thermal Green's functions. In section 3 we describe the gravity solution, and in section 4 we review the Minkowski prescription used to compute the Green's functions from gravity. In section 5 we compute the correlators of R-charge currents and show the emergence of the pole corresponding to R-charge diffusion. As a by-product we find the R-charge diffusion rate. In section 6 we discuss the hydrodynamic shear mode; as in the case with the R-charge currents, this mode also has a diffusive pole. Section 7 contains concluding remarks.

## 2. Hydrodynamic preliminaries

To see the constraints imposed by hydrodynamics on thermal correlation functions, let us consider some theory, at finite temperature, with a conserved global charge. We denote the corresponding current as $j^\mu$, so $j^0$ is the spatial density of the charge. We assume zero chemical potential for this charge, so in thermal equilibrium $\langle j^0 \rangle = 0$. The quantity of interest is the retarded thermal Green's function

$$G^R_{\mu\nu}(\omega, \boldsymbol{q}) = -i \int d^4x\, e^{-iq\cdot x}\, \theta(t) \langle [j_\mu(x), j_\nu(0)] \rangle \,, \tag{2.1}$$

where $q = (\omega, \boldsymbol{q})$, $x = (t, \boldsymbol{x})$. This function determines the response of the system on a small external source coupled to the current. When $\omega$ and $\boldsymbol{q}$ are small, the external perturbation varies slowly in space and time, and a macroscopic hydrodynamic description for its evolution is possible. For example, $j^0$ evolves according to the diffusion equation (Fick's law [12])

$$\partial_0 j^0 = D\nabla^2 j^0 \,, \tag{2.2}$$

where $D$ is a diffusion constant with dimension of length. This equation corresponds to an overdamped mode, whose dispersion relation is

$$\omega = -iDq^2 \,, \tag{2.3}$$

which implies that there has to be a pole, located at $\omega = -iDq^2$ in the complex $\omega$-plane, in the retarded correlation functions of $j^0$ [10].

As a slightly more complicated example, let us consider the correlators of the components of the stress-energy tensor $T^{\mu\nu}$. The linearized hydrodynamic equations have the form

$$\begin{aligned} \partial_0 \tilde{T}^{00} + \partial_i T^{0i} &= 0 \,, \\ \partial_0 T^{0i} + \partial_j \tilde{T}^{ij} &= 0 \,, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \tilde{T}^{00} &= T^{00} - \epsilon \,, \qquad\qquad \epsilon = \langle T^{00} \rangle \,, \\ \tilde{T}^{ij} &= T^{ij} - P\delta^{ij} = -\frac{1}{\epsilon + P}\left[\eta\left(\partial_i T^{0j} + \partial_j T^{0i} - \frac{2}{3}\delta^{ij}\partial_k T^{0k}\right) + \zeta\delta^{ij}\partial_k T^{0k}\right], \end{aligned} \tag{2.5}$$

$\epsilon$ and $P$ are the energy density and pressure, $\eta$ and $\zeta$ are the shear and bulk viscosities, respectively. eqs. (2.4) possess two types of eigenmodes: the shear modes, which are the transverse fluctuations of the momentum density $T^{0i}$, with a purely imaginary eigenvalue

$$\omega = -\frac{i\eta}{\epsilon + P} q^2 \,, \tag{2.6}$$

and the sound wave, which is the simultaneous fluctuation of the energy density $T^{00}$ and the longitudinal component of the momentum density $T^{0i}$, with the dispersion relation

$$\omega = u_s q - \frac{i}{2}\frac{1}{\epsilon + P}\left(\zeta + \frac{4}{3}\eta\right) q^2 \,, \qquad u_s^2 = \frac{\partial P}{\partial \epsilon} \,. \tag{2.7}$$

The poles (2.3) and (2.6) will be reproduced from gravity calculations, while (2.7) is deferred to future work. If the theory is conformal, then the stress-energy tensor is traceless, so $\epsilon = 3P$ and $\zeta = 0$.

## 3. Gravity preliminaries

The non-extremal three-brane background is a solution of the type IIB low energy equations of motion,

$$R_{MN} = \frac{1}{96} F_{MPQRS} F_N^{\ PQRS} \,, \tag{3.1a}$$

$$F_{(5)} = *F_{(5)} \,, \tag{3.1b}$$

where all other supergravity fields are consistently set to zero. We use notations $M, N, \ldots$ for the ten-dimensional indices, $\mu, \nu, \ldots$ for the five-dimensional and $i, j, \ldots$ for the four-dimensional ones. The solution is given by the metric

$$ds_{10}^2 = H^{-1/2}(r)\left[-f dt^2 + dx^2 + dy^2 + dz^2\right] + H^{1/2}(r)\left(f^{-1}dr^2 + r^2 d\Omega_5^2\right), \tag{3.2}$$

where $H(r) = 1 + R^4/r^4$, $f(r) = 1 - r_0^4/r^4$, and the Ramond-Ramond five-form,

$$F_5 = -\frac{4R^2}{H^2 r^5}(R^4 + r_0^4)^{1/2}(1 + *)\, dt \wedge dx \wedge dy \wedge dz \wedge dr\,, \tag{3.3}$$

with all other fields vanishing.

In the near-horizon limit $r \ll R$ the metric becomes

$$ds_{10}^2 = \frac{(\pi T R)^2}{u}\left(-f(u)dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2\,, \tag{3.4}$$

where $T = r_0/\pi R^2$ is the Hawking temperature, and we have introduced $u = r_0^2/r^2$ and $f(u) = 1 - u^2$. The horizon corresponds to $u = 1$, the spatial infinity to $u = 0$.

In the language of the gauge theory/gravity correspondence, the background (3.4) with the non-extremality parameter $r_0$ is dual to the $\mathcal{N} = 4$ SU($N$) SYM at finite temperature $T = r_0/\pi R^2$ in the limit of $N \to \infty$, $g_{YM}^2 N \to \infty$.

## 4. Prescription for minkowskian correlators

From the previous discussion, one notes that in order to see the emergence of hydrodynamic behavior, one needs to compute thermal Green's functions in *Minkowski* space. In ref. [16] we formulate and discuss in detail a prescription for computing two-point Green's functions from gravity. For convenience this prescription is given here; technical details can be found in ref. [16].

First let us recall that in euclidean space, the gravity/gauge theory duality is encoded in the following equality,

$$\langle e^{\int_{\partial M} \phi_0 \hat{\mathcal{O}}} \rangle = e^{-S_{\rm cl}[\phi_0]}\,, \tag{4.1}$$

where $\hat{\mathcal{O}}$ is some boundary CFT operator and $\phi$ is the bulk field which couples to it.

We do not claim to have a full minkowskian analog of eq. (4.1). Rather, we shall concentrate on two-point functions, and formulate our prescription specifically for those functions. In euclidean space, eq. (4.1) implies that finding $\langle \hat{\mathcal{O}}(x)\hat{\mathcal{O}}(0)\rangle$ amounts to computing the second functional derivative of $S_{\rm cl}$ on the boundary value $\phi_0$. It can be shown that the computation of $\langle \hat{\mathcal{O}}\hat{\mathcal{O}}\rangle$ is reduced to the following three steps. We shall denote the radial coordinate as $u$, and for definiteness assume that the boundary is located at $u = 0$, and the horizon is at some positive $u$.

i) From the classical action for $\phi$ one extracts the function $A(u)$ staying in front of $(\partial_u \phi)^2$ in the kinetic term,

$$S_{\rm cl} = \frac{1}{2}\int du\, d^4x\, A(u)(\partial_u \phi)^2 + \cdots \tag{4.2}$$

ii) Solving the linearized field equation for $\phi$, one expresses the bulk field $\phi$ via its value of $\phi_0$ at the boundary,

$$\phi(u, q) = f_q(u)\phi_0(q)\,, \tag{4.3}$$

where we work in momentum space. By definition, the mode function $f_q(u)$ is equal to 1 at $u = 0$.

iii) The euclidean Green's function is then

$$G_E(q) = -A(u) f_{-q}(u)\partial_u f_q(u)|_{u\to 0}\,. \tag{4.4}$$

One can see how these three steps work by recalling that the classical action (4.2), for classical solutions, reduces to the boundary term $\sim A\phi\phi'$. Note that in taking the limit $u \to 0$ in eq. (4.4) one may need to throw away the contact terms.

Now we can proceed with the formulation of the prescription for minkowskian correlators.

i) The same as in the euclidean case.

ii) In Minkowski space one has to specify the boundary condition at the horizon in addition to that at the boundary $u = 0$. We impose the incoming-wave boundary condition (waves are only absorbed by the black branes but not emitted from there) for all Fourier components $\phi_q$ with timelike $q$. For spacelike $q$'s, we require regularity at the horizon.

iii) The retarded thermal Green's function is

$$G^R(q) = A(u) f_{-q}(u)\partial_u f_q(u)|_{u\to 0}\,. \tag{4.5}$$

Choosing the outgoing-wave condition at the horizon would yield the advanced Green's function $G^A$ instead. The sign in eq. (4.5) corresponds to the standard convention of the retarded and advanced Green's functions,

$$\begin{aligned} G^R(\omega, \boldsymbol{q}) &= -i\int d^4x\, e^{-iq\cdot x}\,\theta(t)\langle[\hat{\mathcal{O}}(x),\, \hat{\mathcal{O}}(0)]\rangle\,,\\ G^A(\omega, \boldsymbol{q}) &= i\int d^4x\, e^{-iq\cdot x}\,\theta(-t)\langle[\hat{\mathcal{O}}(x),\, \hat{\mathcal{O}}(0)]\rangle\,. \end{aligned} \tag{4.6}$$

In ref. [16] we verify that the three steps outlined above indeed give the correct retarded Green's functions in several cases where independent verification is possible. Admittedly, this three-step prescription is aesthetically unsatisfactory: it cannot be formulated as succinctly as eq. (4.1). Nevertheless, it does seem to work. One can hope that our prescription can be embedded in future general framework which allows the calculation of higher-point Green's functions as well. Despite the shortcomings, the prescription at hand is sufficient for the purpose of this paper.



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## 5. Thermal R-current correlators in $\mathcal{N}=4$ SYM and R-charge diffusion

To compute the current correlators, we use an approach similar to the one taken at zero temperature [13, 14], the only difference being that we work in Minkowski rather than in the euclidean space and use the *non*-extremal supergravity background. Our starting point is the five-dimensional Maxwell action in the background (3.4),

$$S = -\frac{1}{4g_{SG}^2}\int d^5x\sqrt{-g}\,F^a_{\mu\nu}F^{\mu\nu\;a}\,, \tag{5.1}$$

where $g_{SG} = 4\pi/N$ is the effective coupling constant fixed in [13]. Reinstating powers of $R$, we have $g_{SG}^2 = 16\pi^2 R/N^2$.

We shall work in the gauge $A_u = 0$. After imposing this gauge condition, one still can make the (residual) gauge transformations $A_\mu \to A_\mu + \partial_\mu\Lambda$, where $\Lambda$ is independent of $u$. We use the Fourier decomposition

$$A_i = \int \frac{d^4q}{(2\pi)^4} e^{-i\omega t + i\boldsymbol{q}\cdot\boldsymbol{x}} A_i(q,u)\,. \tag{5.2}$$

To simplify calculations, we also choose $\boldsymbol{q}$ along the $z$-direction on the brane, so the four-momentum is $q=(\omega,0,0,q)$. Defining the dimensionless energy and momentum,

$$\mathfrak{w} = \frac{\omega}{2\pi T}\,, \qquad \mathfrak{q} = \frac{q}{2\pi T}\,, \tag{5.3}$$

the five-dimensional Maxwell equations,

$$\frac{1}{\sqrt{-g}}\partial_\nu[\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)] = 0\,, \tag{5.4}$$

reduce to the following set of the ordinary differential equations

$$\mathfrak{w}A_t' + \mathfrak{q}f\,A_z' = 0\,, \tag{5.5a}$$

$$A_t'' - \frac{1}{uf}\left(\mathfrak{q}^2 A_t + \mathfrak{w}\mathfrak{q}A_z\right) = 0\,, \tag{5.5b}$$

$$A_z'' + \frac{f'}{f}A_z' + \frac{1}{uf^2}\left(\mathfrak{w}^2 A_z + \mathfrak{w}\mathfrak{q}A_t\right) = 0\,, \tag{5.5c}$$

$$A_\alpha'' + \frac{f'}{f}A_\alpha' + \frac{1}{uf}\left(\frac{\mathfrak{w}^2}{f} - \mathfrak{q}^2\right)A_\alpha = 0\,, \tag{5.5d}$$

where $\alpha$ stands for either $x$ or $y$, and the prime denotes the derivative with respect to $u$. Not all these equations are independent: eqs. (5.5a) and (5.5b) imply (5.5c). One can also check that they are invariant under residual gauge transformations $A_t \to A_t - \omega\Lambda$, $A_z \to A_z + q\Lambda$.

One sees that $A_t$ and $A_z$ satisfy a coupled system of equations, while each component of $A_\alpha$ satisfies a stand-alone equation. The R-charge diffusion appears only in the sector of $A_t$ and $A_z$. One can reduce eqs. (5.5a)–(5.5c) to a single equation by expressing, from eq. (5.5b), $A_z$ in terms of $A_t$,

$$A_z = \frac{uf}{\mathfrak{w}\mathfrak{q}}A_t'' - \frac{\mathfrak{q}}{\mathfrak{w}}A_t\,, \tag{5.6}$$

and then substituting $A_z$ into eq. (5.5a). One obtains

$$A_t''' + \frac{(uf)'}{uf} A_t'' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f(u)}{uf^2} A_t' = 0\,, \tag{5.7}$$

Thus eqs. (5.5d), (5.6) and (5.7) together with the boundary conditions determine all components of $A_i$ (up to a residual gauge transformation).

Equation (5.7) is a second-order differential equation for $A_t'$ in which $u = 1$ is a singular point. According to the general technique for solving such equations, one first has to determine the singular behavior of $A_t$. If one substitutes into eq. (5.7) $A_t' = (1-u)^\nu F(u)$, where $F(u)$ is a regular function, one finds that only two values of $\alpha$ are allowed: $\nu_\pm^{(1)} = \pm i\mathfrak{w}/2$. The "incoming wave" boundary condition at the horizon singles out $\nu_-$. The equation for $F(u)$ has the form

$$F'' + \left(\frac{1-3u^2}{uf} + \frac{i\mathfrak{w}}{1-u}\right) F' + \frac{i\mathfrak{w}(1+2u)}{2uf} F + \frac{\mathfrak{w}^2[4 - u(1+u)^2]}{4uf^2} F - \frac{\mathfrak{q}^2}{uf} F = 0\,. \tag{5.8}$$

In the long-wavelength, low-frequency limit, $\mathfrak{w}$ and $\mathfrak{q}$ are small parameters, and the solution to eq. (5.8) can be obtained perturbatively as a double series in $\mathfrak{w}$ and $\mathfrak{q}^2$ (cf. [15]),

$$F(u) = F_0 + \mathfrak{w} F_1 + \mathfrak{q}^2 G_1 + \mathfrak{w}^2 F_2 + \mathfrak{w}\mathfrak{q}^2 H_1 + \mathfrak{q}^4 G_2 + \cdots\,. \tag{5.9}$$

The explicit form for the first three terms, essential for obtaining the leading-order expression for the correlators, is rather simple,

$$F_0 = C\,, \qquad F_1 = \frac{iC}{2} \ln \frac{2u^2}{1+u}\,, \qquad G_1 = C \ln \frac{1+u}{2u}\,. \tag{5.10}$$

The functions $F_2$, $G_2$ and $H_1$ are written explicitly in appendix A.

We now substitute our solution for $A_t'$ into eq. (5.6) and take the limit $u \to 0$ assuming the boundary conditions

$$\lim_{u\to 0} A_t(u) = A_t^0\,, \qquad \lim_{u \to 0} A_z(u) = A_z^0\,. \tag{5.11}$$

This determines the constant $C$ in terms of $A_t^0$, $A_z^0$:

$$C = \frac{\mathfrak{q}^2 A_t^0 + \mathfrak{w}\mathfrak{q} A_z^0}{Q(\mathfrak{w}, \mathfrak{q})}\,, \tag{5.12}$$

where $Q(\mathfrak{w}, \mathfrak{q})$ has the following expansion over the small arguments,

$$Q(\mathfrak{w}, \mathfrak{q}) = i\mathfrak{w} - \mathfrak{q}^2 + O(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2, \mathfrak{q}^4)\,. \tag{5.13}$$

$C$ is obviously invariant under the residual gauge transformations. Having thus found $A_t'(u)$, we can obtain $A_z'(u)$ using eq. (5.5a).

Similarly, the solution of eq. (5.5d) is found to be

$$\begin{aligned} A_\alpha = & \frac{8\,A_a^0 (1-u)^{-i\mathfrak{w}/2}}{8 - 2i\mathfrak{w}\ln 2 + \pi^2 \mathfrak{q}^2} \times \\ & \times \left[1 + \frac{i\mathfrak{w}}{2} \ln \frac{1+u}{2} + \frac{\mathfrak{q}^2}{2}\left(\frac{\pi^2}{12} + \mathrm{Li}_2(-u) + \ln u \ln(1+u) + \mathrm{Li}_2(1-u)\right)\right] + O\left(\mathfrak{w}^2, \mathfrak{q}^4, \mathfrak{w}\mathfrak{q}^2\right)\,. \end{aligned} \tag{5.14}$$

Near the horizon, where $u = \epsilon$ is small, the solutions we found imply the following relation between the radial derivatives of the fields and their boundary values,

$$A'_\alpha = i\omega A^0_\alpha \,, \tag{5.15a}$$

$$A'_t = (\mathfrak{q}^2 A^0_t + \mathfrak{w}\mathfrak{q} A^0_z) \ln \epsilon + \frac{\mathfrak{q}^2 A^0_t + \mathfrak{w}\mathfrak{q} A^0_z}{i\mathfrak{w} - \mathfrak{q}^2} \,, \tag{5.15b}$$

$$A'_z = -(\mathfrak{w}\mathfrak{q} A^0_t + \mathfrak{w}^2 A^0_z) \ln \epsilon - \frac{\mathfrak{w}\mathfrak{q} A^0_t + \mathfrak{w}^2 A^0_z}{i\mathfrak{w} - \mathfrak{q}^2} \,. \tag{5.15c}$$

On the other hand, the terms in the action which contain two derivatives with respect to $u$ are

$$\begin{aligned} S &= -\frac{N^2}{32\pi^2 R} \int du\, d^4x \, \sqrt{-g}\, g^{uu} g^{ij} \partial_u A_i \partial_u A_j + \cdots \\ &= \frac{N^2 T^2}{16} \int du\, d^4x \, [A'^2_t - f(A'^2_x + A'^2_y + A'^2_z)] + \cdots \end{aligned} \tag{5.16}$$

Applying the prescription formulated in section 4, one finds

$$G^{ab}_{xx} = G^{ab}_{yy} = -\frac{iN^2 T\omega\, \delta^{ab}}{16\pi} + \cdots \,, \tag{5.17a}$$

$$G^{ab}_{tt} = \frac{N^2 T q^2\, \delta^{ab}}{16\pi(i\omega - Dq^2)} + \cdots \,, \tag{5.17b}$$

$$G^{ab}_{tz} = G^{ab}_{zt} = -\frac{N^2 T\omega q\, \delta^{ab}}{16\pi(i\omega - Dq^2)} + \cdots \,, \tag{5.17c}$$

$$G^{ab}_{zz} = \frac{N^2 T\omega^2\, \delta^{ab}}{16\pi(i\omega - Dq^2)} + \cdots \,, \tag{5.17d}$$

where $\cdots$ denotes corrections of order $\mathfrak{w}^2$, $\mathfrak{w}\mathfrak{q}^2$ or $\mathfrak{q}^4$, and

$$D = \frac{1}{2\pi T} \,. \tag{5.18}$$

We see that the correlation functions of $j^0$ and $j^z$ contains the diffusion pole expected from hydrodynamic arguments. In contrast, those of $j^x$ and $j^y$ do not have this pole. The constant $D$ is the diffusion constant of R-charges; its value in the large 't Hooft coupling limit $g^2_{YM} N \to \infty$ is found explicitly in eq. (5.18). This result can be regarded as a nontrivial prediction for the strongly coupled $\mathcal{N} = 4$ SYM at finite temperature. One observes that in this limit $D$ is independent of the 't Hooft coupling, sharing this property with the shear viscosity ([17], see also below). Also, the diffusion constant does not contain a power of $N$. It is amazing that it is at all possible to compute a kinetic coefficient in a strongly coupled theory, and the final result is as simple as eq. (5.18). What is also interesting to note is that the calculation of $D$ above is, arguably, technically simpler than similar calculations in weakly-coupled field theories [18]–[21].

It is interesting to compare eq. (5.18) with the result at weak coupling. In this regime, the diffusion constant is proportional to the mean free path, which implies

$$D \sim \frac{1}{(g^2_{\rm YM} N)^2 T \ln \frac{1}{(g^2_{\rm YM} N)}} \,, \qquad g^2_{\rm YM} N \ll 1 \,. \tag{5.19}$$



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Equations (5.18) and (5.19) suggest that the behavior of $D$ as a function of the 't Hooft coupling $g_{\rm YM}^2 N$ is

$$D = f_D(g_{\rm YM}^2 N)\frac{1}{T}\,, \tag{5.20}$$

where $f_D(x) \sim x^{-2}\ln^{-1}(1/x)$ for $x \ll 1$ and $f_D(x) = 1/(2\pi)$ for $x \gg 1$.

With some extra effort, one can also compute corrections to these results in the parameters $\mathfrak{w}$ and $\mathfrak{q}$. To the next order in perturbation theory, the correlators which have the pole are given by

$$G_{tt}^{ab} = \frac{N^2T^2\mathfrak{q}^2 P(\mathfrak{w},\mathfrak{q})\;\delta^{ab}}{8\,Q(\mathfrak{w},\mathfrak{q})}\,, \tag{5.21a}$$

$$G_{tz}^{ab} = G_{zt}^{ab} = -\frac{N^2T^2\mathfrak{w}\mathfrak{q}P(\mathfrak{w},\mathfrak{q})\;\delta^{ab}}{8\,Q(\mathfrak{w},\mathfrak{q})}\,, \tag{5.21b}$$

$$G_{zz}^{ab} = \frac{N^2T^2\mathfrak{w}^2 P(\mathfrak{w},\mathfrak{q})\;\delta^{ab}}{8\,Q(\mathfrak{w},\mathfrak{q})}\,, \tag{5.21c}$$

where corrections of order $\mathcal{O}(\mathfrak{w}^3, \mathfrak{w}^2\mathfrak{q}^2, \mathfrak{w}\mathfrak{q}^4, \mathfrak{q}^6)$ and higher are omitted, and

$$\begin{aligned} P(\mathfrak{w},\mathfrak{q}) &= 1 + \ln 2\left(\frac{i}{2}\mathfrak{w} - \mathfrak{q}^2\right),\\ Q(\mathfrak{w},\mathfrak{q}) &= i\mathfrak{w} - \mathfrak{q}^2 + \ln 2\left(\frac{\mathfrak{w}^2}{2} + \frac{i}{2}\mathfrak{w}\mathfrak{q}^2 - \mathfrak{q}^4\right). \end{aligned} \tag{5.22}$$

The zeroes of $Q(\mathfrak{w},\mathfrak{q})$ determine the poles of the correlators. For small $\mathfrak{q}$ only one of the zeros has a value compatible with our assumption $\mathfrak{w} \ll 1$, others must be discarded. The position of the pole is given by

$$\mathfrak{w} = -i\mathfrak{q}^2(1 + \mathfrak{q}^2\ln 2)\,, \tag{5.23}$$

or, in terms of the dimensionful energy and momentum $\omega$, $q$,

$$i\omega = Dq^2\left(1 + \frac{q^2\ln 2}{4\pi^2T^2}\right). \tag{5.24}$$

## 6. Near-extremal metric perturbations and the shear mode

To compute the two-point function of the stress-energy tensor in the boundary theory, we consider a small perturbation of the five-dimensional part of the near-extremal background (3.4), $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$, where $g_{\mu\nu}^{(0)}$ is given by

$$ds_5^2 = \frac{\pi^2T^2R^2}{u}\left(-f(u)dt^2 + d\boldsymbol{x}^2\right) + \frac{R^2}{4f(u)u^2}du^2\,. \tag{6.1}$$

The part of the ten-dimensional metric corresponding to $\mathrm{S}^5$ remains unperturbed, and the five-form field (3.3) is unchanged to the first order in perturbation. The perturbed metric satisfies

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\nu}^{(0)} + \mathcal{R}_{\mu\nu}^{(1)} + \cdots = \frac{2\Lambda}{3}g_{\mu\nu}\,, \tag{6.2}$$

where $\Lambda = -6/R^2$. To the first order in $h_{\mu\nu}$ the Einstein equations are

$$\mathcal{R}^{(1)}_{\mu\nu} = -\frac{4}{R^2} h_{\mu\nu} \,. \tag{6.3}$$

The solution of the Dirichlet problem for eq. (6.3) is then substituted into the $5d$ gravitational action expanded near the background (6.1). The action is given by

$$S = \frac{\pi^3 R^5}{2\kappa_{10}^2} \left[ \int_0^1 du \int d^4x \, \sqrt{-g}\, (\mathcal{R} - 2\Lambda) + 2 \int d^4x \, \sqrt{-h}\, K \right]. \tag{6.4}$$

Here $\kappa_{10} = \sqrt{8\pi G}$ is the ten-dimensional gravitational constant, related to the parameter $R$ of the non-extremal geometry and the number $N$ of coincident branes by $\kappa_{10} = 2\pi^2 \sqrt{\pi} R^4/N$ [5]. The second integral is the Gibbons-Hawking boundary term with $K$ being the trace of the extrinsic curvature of the boundary.

Again, we shall assume the perturbation to be dependent only on $t$ and $z$ (and proportional to $e^{-i\omega t + iqz}$), and choose the gauge where $h_{u\mu} = 0$ for all $\mu$. In this case, the gravitational perturbation can be classified by the spin under the O(2) rotations in the $xy$ plane. Specifically, there are three classes of perturbations (only nonzero components of $h_{\mu\nu}$ are listed):

- $h_{xy} \neq 0$, or $h_{xx} = -h_{yy} \neq 0$;
- $h_{xt}$ and $h_{xz} \neq 0$, or $h_{yt}$ and $h_{yz} \neq 0$;
- $h_{tz}$ and all diagonal elements of $h_{\mu\nu}$ are nonzero, and $h_{xx} = h_{yy}$.

The field equations in each of the classes decouple from the other ones. We shall consider the first two cases. The third case is related to the sound wave in field theory and is not considered in this paper.

### 6.1 Off-diagonal perturbation with $h_{xy} \neq 0$

The simplest case we consider is that when the off-diagonal perturbation $h_{xy} \neq 0$, and all other perturbations vanish. The equations for the perturbation with $h_{xx} = -h_{yy}$ is absolutely identical. eq. (6.3) becomes

$$h''_{xy} + \frac{1-3u^2}{uf} h'_{xy} + \frac{1}{(2\pi T)^2 f^2 u} \left( f \frac{\partial^2 h_{xy}}{\partial z^2} - \frac{\partial^2 h_{xy}}{\partial t^2} \right) - \frac{1+u^2}{fu^2} h_{xy} = 0 \,. \tag{6.5}$$

Introducing a new function $\phi = u h_{xy}/(\pi T R)^2$ (i.e., $\phi = h^x_y$), and using the Fourier component $\phi_k(u)$ defined as in (5.2), we observe that eq. (6.5) becomes an equation for a minimally coupled scalar in the background (6.1):

$$\phi''_k - \frac{1+u^2}{uf} \phi'_k + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2} \phi_k = 0 \,. \tag{6.6}$$

Therefore, the computation now is completely similar to the computation of the Chern-Simon diffusion rate in ref. [16]. The solution representing the incoming wave at the horizon is given by

$$\phi_k(u) = (1-u)^{-i\mathfrak{w}/2} F_k(u) \,, \tag{6.7}$$

where $F_k(u)$ is regular at $u = 1$ and can be written as a series [16]

$$F_k(u) = 1 - \frac{i\mathfrak{w}}{2} \ln \frac{1+u}{2} + \frac{\mathfrak{w}^2}{8} \left[ \left( \ln \frac{1+u}{2} + 8\Big(1 - \frac{\mathfrak{q}^2}{\mathfrak{w}^2}\Big) \right) \ln \frac{1+u}{2} - 4 \, \mathrm{Li}_2 \frac{1-u}{2} \right] + O(\mathfrak{w}^3) \,, \tag{6.8}$$

where $\mathrm{Li}_2(z)$ is the polylogarithm.

The term proportional to $\phi'^2$ in the action is

$$S = -\frac{\pi^2 N^2 T^4}{8} \int du \, d^4x \, \frac{f}{u} \phi'^2 + \cdots \tag{6.9}$$

Let us define, in complete analogy with eq. (2.1), the retarded Green's function for the components of the stress-energy tensor,

$$G_{\mu\nu,\lambda\rho}(\omega, \boldsymbol{q}) = -i \int d^4x \, e^{-iq\cdot x} \, \theta(t) \langle [T_{\mu\nu}(x), \, T_{\lambda\rho}(0)] \rangle \,. \tag{6.10}$$

Then, according to the prescription given in [16], the (retarded) two-point function of the stress-energy tensor (in Fourier space) reads

$$G_{xy,xy}(\omega, \boldsymbol{q}) = -\frac{N^2 T^2}{16} \left( i \, 2\pi T \omega + q^2 \right) . \tag{6.11}$$

This result can be used to determine the shear viscosity of the strongly coupled $\mathcal{N} = 4$ SYM plasma. One has to recall the Kubo formula, which relates the shear viscosity to the correlation function of the stress-energy tensor at zero spatial momentum,

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt \, d\boldsymbol{x} \, e^{i\omega t} \, \langle [T_{xy}(x), \, T_{xy}(0)] \rangle = \frac{\pi}{8} N^2 T^3 \,. \tag{6.12}$$

The result coincides with the one obtained in [17] from the absorption calculation, and also from the value obtained from the pole of the propagator (see below).

### 6.2 Off-diagonal perturbation with $h_{tx} \neq 0$, $h_{xz} \neq 0$

This is the channel where the correlation functions have the diffusion pole. In this case the Einstein equations (6.3) give the following set of coupled equations for the Fourier components of the perturbations $H_t = u h_{tx}/(\pi T R)^2$ and $H_z = u h_{zx}/(\pi T R)^2$:

$$H_t' + \frac{\mathfrak{q} f}{\mathfrak{w}} H_z' = 0 \,, \tag{6.13a}$$

$$H_t'' - \frac{1}{u} H_t' - \frac{\mathfrak{w}\mathfrak{q}}{uf} H_z - \frac{\mathfrak{q}^2}{uf} H_t = 0 \,, \tag{6.13b}$$

$$H_z'' - \frac{1+u^2}{uf} H_z' + \frac{\mathfrak{w}^2}{uf^2} H_z + \frac{\mathfrak{w}\mathfrak{q}}{uf^2} H_t = 0 \,. \tag{6.13c}$$

This system is very similar to the one encountered in the previous section for the potentials $A_t$, $A_z$. Eqs. (6.13a) and (6.13b) imply (6.13c). Solving (6.13b) for $H_z$ we get

$$H_z = \frac{uf}{\mathfrak{w}\mathfrak{q}} H_t'' - \frac{f}{\mathfrak{w}\mathfrak{q}} H_t' - \frac{\mathfrak{q}}{\mathfrak{w}} H_t \,. \tag{6.14}$$

Substituting this into eq. (6.13a) gives

$$H_t''' - \frac{2u}{f} H_t'' + \frac{2uf - \mathfrak{q}^2 f + \mathfrak{w}^2}{uf^2} H_t' = 0\,. \tag{6.15}$$

In the low-frequency, long-wavelength limit, this equation is solved by making the substitution $H_t'(u) = (1-u)^{-i\mathfrak{w}/2} G(u)$ and then solving perturbatively the resulting equation for $G(u)$,

$$G'' - \left(\frac{2u}{f} - \frac{i\mathfrak{w}}{1-u}\right) G' + \frac{1}{f}\left(2 + \frac{i\mathfrak{w}}{2} - \frac{\mathfrak{q}^2}{u} + \frac{\mathfrak{w}^2[4-u(1+u)^2]}{4uf}\right) G = 0\,. \tag{6.16}$$

The solution regular at $u=1$ is

$$G(u) = C\left[u - i\mathfrak{w}\left(1 - u - \frac{u}{2}\ln\frac{1+u}{2}\right) + \frac{\mathfrak{q}^2(1-u)}{2}\right] + O\left(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2, \mathfrak{q}^4\right). \tag{6.17}$$

Taking the limit $u \to 0$ in the eq. (6.13b) we find $C$ in terms of the boundary values $H_t^0$ and $H_z^0$:

$$C = \frac{\mathfrak{q}^2 H_t^0 + \mathfrak{q}\mathfrak{w} H_z^0}{i\mathfrak{w} - \frac{\mathfrak{q}^2}{2}}\,. \tag{6.18}$$

The terms proportional to $H_t'^2$ and $H_z'^2$ in the action (6.4) are

$$S = -\frac{\pi^2 N^2 T^4}{8} \int du\, d^4x\, \frac{1}{u}\left(-H_t'^2 + f H_z'^2\right) + \cdots \tag{6.19}$$

Again, as in the previous section, all constant and contact terms are ignored. Computing the correlators, we get

$$G_{tx,tx}(\omega, \boldsymbol{q}) = \frac{N^2 \pi T^3 q^2}{8(i\omega - \mathcal{D}q^2)}\,, \tag{6.20a}$$

$$G_{tx,xz}(\omega, \boldsymbol{q}) = -\frac{N^2 \pi T^3 \omega q}{8(i\omega - \mathcal{D}q^2)}\,, \tag{6.20b}$$

$$G_{xz,xz}(\omega, \boldsymbol{q}) = \frac{N^2 \pi T^3 \omega^2}{8(i\omega - \mathcal{D}q^2)}\,, \tag{6.20c}$$

where the formulas are valid up to corrections of order $\mathcal{O}(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2, \mathfrak{q}^4)$, and

$$\mathcal{D} = \frac{1}{4\pi T}\,, \tag{6.21}$$

i.e., is twice smaller than the diffusion constant for the R-charge.

According to formula (2.6), $\mathcal{D} = \eta/(\epsilon + P)$. The pressure $P$ can be found from the expression for the entropy density [5], which differs from that at zero gauge coupling by a well-known factor of 3/4,

$$s = \frac{3}{4}s_0 = \frac{\pi^2}{2}N^2T^3\,, \tag{6.22}$$

and the thermodynamic relation $s = \partial P/\partial T$. Moreover, $\epsilon = 3P$, thus $\epsilon + P = \frac{\pi^2}{2}N^2T^4$. Equations (2.6) and (6.21) then yield $\eta = \frac{\pi}{8}N^2T^3$, which agrees with ref. [17] and eq. (6.12). Thus, we have seen the consistency of several seemingly unrelated calculations: the calculation of the entropy, the calculation of the viscosity via absorption, the calculation of the correlators (6.20), and the determination of the pole from linearized hydrodynamics. In particular, if the entropy differed from the free-gas value by a factor other than 3/4, such a consistency would not have been seen.

Again, the comparison of our result for $\eta$ with that in the weak-coupling regime [18]–[21] suggests that the behavior of $\eta$ as a function of the 't Hooft coupling is

$$\eta = f_\eta(g^2_{\mathrm{YM}}N)N^2T^3 \tag{6.23}$$

where $f_\eta(x) \sim x^{-2}\ln^{-1}(1/x)$ for $x \ll 1$ and $f_\eta(x) = \frac{\pi}{8}$ for $x \gg 1$. Equations (5.20) and (6.23) suggest that the $\alpha'$ corrections to $D$ and $\eta$ are positive.

## 7. Conclusion

In this paper, we have used a Minkowski AdS/CFT prescription to compute the real-time Green's functions of the R-charge currents and the components of the stress-energy tensor in the $\mathcal{N} = 4$ SYM theory at strong coupling and finite temperature. We showed that the results agree with the expectations from hydrodynamic theory, which can be interpreted as a verification of the Minkowski prescription, or of the existence of the hydrodynamic behavior in the finite-temperature field theory, or both. A more general conclusion is that our results support the validity of the finite-temperature AdS/CFT correspondence, even in four dimensions where no independent checks based on non-renormalization arguments were known to exist. We hope that the calculations performed in this paper can be extended to cover the sound wave, as well as to other examples of gravity/gauge theory duality.

## Acknowledgments

The authors thank I.R. Klebanov, P.K. Kovtun, M. Porrati, and L.G. Yaffe for discussions, and C.P. Herzog for his comments on the manuscript. This work is supported, in part, by DOE Grant No. DOE-ER-41132. The work of D.T.S. is supported, in part, by the Alfred P. Sloan Foundation. G.P. is supported in part by C.N.R.–Italy Grant.



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## A. Perturbative solution for $A_t'(u)$

Here we give explicit expressions for the first few terms of the perturbative expansion (5.9). Integration constants are fixed by requiring functions $F_1$, $F_2$, $G_1$, $G_2$, $H_1$ to be regular at the horizon ($u = 1$), and to have a vanishing limit as $u \to 1$.

$$\begin{aligned} F_2 &= \frac{C}{24}\Big[\pi^2 + 3\ln^2 2 + 3\ln^2(1+u) + 6\ln 2 \ln\frac{u^2}{1+u} + \\ &\quad + 12\ \mathrm{Li}_2(1-u) + 12\ \mathrm{Li}_2(-u) - 12\ \mathrm{Li}_2\ \frac{1-u}{2}\Big], \end{aligned} \tag{A.1}$$

$$\begin{aligned} H_1 &= iC\Big[-\frac{\pi^2}{12} - i\pi\ln 2 + \ln^2 2 + \frac{1}{2}\ln(1+u)\ln\frac{u(1-u)^2}{4} + \\ &\quad + \frac{\ln 2}{2}\ln\frac{u}{(u-1)^2} + \mathrm{Li}_2(-u) + \mathrm{Li}_2\,(1-u) + \mathrm{Li}_2\ \frac{1+u}{2}\Big], \end{aligned} \tag{A.2}$$

$$\begin{aligned} G_2 &= C\Big[-\frac{\pi^2}{24} + \frac{1}{2}\ln^2 u - \frac{1}{2}\ln u\ln(1+u) - \ln\left(-\frac{1}{u}\right)\ln[2u(1+u)] + \\ &\quad + \mathrm{Li}_2(2) - \mathrm{Li}_2\left(1+\frac{1}{u}\right) + \mathrm{Li}_2\ \frac{1-u}{2} - \frac{1}{2}[\mathrm{Li}_2(1-u) + \mathrm{Li}_2(-u)]\Big]. \end{aligned} \tag{A.3}$$

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